Section 3.4
Systems of Equations in 3 Variables

- Visualize a System of 2 Equations of 2 variables: $Ax + By = C$
  - Each equation defines a straight line on the flat plane of an $x$-$y$ coordinate graph
  - Usually these lines cross at a single $(x,y)$ point which we call the solution
  - We can use Substitution or Elimination to solve for this single point solution

- Visualize a System of 3 equations in 3 variables: $Ax + By + Cz = D$
  - Each equation defines a flat plane that can be graphed on a 3-D $x$-$y$-$z$ graph
  - Usually these planes cross at a single point $(x,y,z)$ which we call the solution
  - Another type of solution has an infinite number of points: a 3-dimensional straight line
  - We can use repeated Elimination or Substitution to solve for single point solutions

- Some systems, such as parallel or triangular planes, do not have a solution
How to Solve
Systems of Equations in 3 Variables

- **Review:** Procedure to solve Systems of 2 Equations of 2 variables:
  - Put both equations into simplified Standard Form \( Ax + By = C \)
  - Eliminate a variable:
    - Combine the equations to create a new equation in 1 variable
  - Solve the new equation for the remaining variable
  - Substitute that value into either original equation to solve for the other variable
  - Check the values in the other original equation
  - Present the solution as an ordered pair \((x, y)\)

- Procedure to solve Systems of 3 Equations of 3 variables:
  - Put all 3 equations into simplified Standard Form \( Ax + By + Cz = D \)
  - Eliminate a variable:
    - Combine 2 pairs of equations to create two new equations in 2 variables
  - Solve the new system of 2 equations (**above**), to solve for 2 variables
  - Substitute those values into any original equation to solve for the 3\(^{rd}\) variable
  - Check the values in the other two original equations
  - Present the solution as an ordered triple \((x, y, z)\)

- A **Solution Point** must make all 3 equations true
- When plotted on a 3-D graph, the solution is the point where the 3 planes intersect. If no point solution exists, it is either dependent or inconsistent.
The definition of a linear equation can be extended to include equations of the form

$$Ax + By + Cz = D.$$ 

The solution of a system of three linear equations with three variables is an **ordered triple** of numbers.

**FIGURE 3-10**

Consistent system

- The three planes intersect at a single point $P$: One solution

Consistent system

- The three planes have a line $l$ in common: Infinitely many solutions

Inconsistent systems

- The three planes have no point in common: No solutions
Solving a Consistent System

- Solve the system.

(A) \[2x - y + 3z = 6\]
(B) \[3x - 5y + 4z = 7\]
(C) \[2x + y + z = -2\]

Three ways to pick pairs: (Why eliminate \(y\) ?)
- (A) and (B)
- (A) and (C)
- (B) and (C)
Solution to

\[
\begin{align*}
2x - y + 3z &= 6 \\
3x - 5y + 4z &= 7 \\
2x + y + z &= -2
\end{align*}
\]

- Adding (A) and (C) will eliminate \( y \)
  
  (A) \( 2x - y + 3z = 6 \)
  
  (C) \( 2x + y + z = -2 \)

  (D) \( 4x + 4z = 4 \) \quad \text{first new equation in 2 variables}

- Adding (B) and \( 5 \cdot (C) \) will also eliminate \( y \)
  
  (B) \( 3x - 5y + 4z = 7 \)
  
  5 \cdot (C) \( 10x + 5y + 5z = -10 \)

  (E) \( 13x + 9z = -3 \) \quad \text{second new equation in 2 variables}

- Solve (D) and (E) like a system of two equations (next page)
  - Use Substitution or Addition
Solution to

\[
\begin{aligned}
2x - y + 3z &= 6 \\
3x - 5y + 4z &= 7 \\
2x + y + z &= -2
\end{aligned}
\] (D)

\[
4x + 4z = 4
\] (E)

\[
13x + 9z = -3
\] (E)

\[
\begin{align*}
\square & \quad \text{Well use substitution of } x \text{ from (D) into (E) to find } z \\
& \quad (D) \quad 4x + 4z = 4 \\
& \quad (D_1) \quad x = 1 - z \quad \text{move } 4z \text{ to the other side, divide by 4} \\
\square & \quad \text{Substitute } x \text{ from (D}_1\text{) into (E)} \\
& \quad (E) \quad 13x + 9z = -3 \\
& \quad (D_1 \to E) \quad 13(1 - z) + 9z = -3 \\
& \quad 13 - 13z + 9z = -3 \quad \text{use distribution, then simplify} \\
& \quad -4z = -16 \quad \Rightarrow \quad z = 4 \\
\square & \quad \text{Substitute } z \text{ into (D) or (E) or (D}_1\text{) to find } x \\
& \quad (z=4 \to D) \quad 4x + 4(4) = 4 \\
& \quad 4x + 16 = 4 \\
& \quad 4x = -12 \quad \Rightarrow \quad x = -3 \\
\square & \quad \text{Substitute } x \text{ and } z \text{ into (A) or (B) or (C) to find } y \\
& \quad (x=-3,z=4 \to C) \quad 2(-3) + y + (4) = -2 \\
& \quad -6 + y + 4 = -2 \\
& \quad y = 0 \quad ^{\text{3.4}} \\
& \quad \text{Solution is } (-3, 0, 4)
Solve this system

(A) \(4x - 2y - 3z = 5\)
(B) \(-8x - y + z = -5\)
(C) \(2x + y + 2z = 5\)
Dependent and Inconsistent Systems

**FIGURE 3-11**

(a) When three planes coincide, the equations are dependent, and there are infinitely many solutions.

(b) When three planes intersect in a common line, the equations are dependent, and there are infinitely many solutions.

(c) When two planes coincide and are parallel to a third plane, the system is inconsistent, and there are no solutions.
Inconsistent System

- Solve the system.
  (A) $2a + b - 3c = 8$
  (B) $3a - 2b + 4c = 10$
  (C) $4a + 2b - 6c = -5$

- Let’s eliminate the variable $b$
- (B) + (C) sums to (D) $7a - 2c = 5$
- $-2(A) + (C)$ reduces to $0 = -21$ which is *inconsistent*
- (A) and (C) are equations of parallel planes and (B) intersects both of them
- Therefore, this system has *no solution*
Dependent Systems

Try to solve this system

(A) \(3x - 2y + z = -1\)
(B) \(2x + y - z = 5\)
(C) \(5x - y = 4\)

Adding (A) and (B) yields (D), which is identical to (C)
(D) \(5x - y = 4\)

Subtracting (D) from (C) yields \(0 = 0\), proof of a Dependent System
Arranging in Simplified Standard Form -
What do you do when:

- **Equation terms need rearranging:**
  - $x + y + z = 83$
  - $y = 2x + 3z - 2$
  - $z = x - y$

- **Equations have missing terms:**
  - $x + z = 0$
  - $x + y = 3$
  - $y + z = 2$

- **Equation coefficients are decimals or fractions:**
  - $x - \frac{1}{2}y + z = 3$
  - $\frac{3}{4}x + y + \frac{2}{3}z = 2$
  - $0.5x + 1.2y - 0.33z = 1$
What Next?

- Section 4.1 – Inequalities and Applications